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Normal generation of line bundles on multiple coverings ☆

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ABSTRACT

A classical result says that a line bundle $\mathcal{L}$ on a smooth curve C of genus g with $\deg \mathcal{L} \geqslant 2g+1$ is normally generated. However, it is known that several line bundles of degree d which fail to be normally generated appear on curves of genus g, which are multiple coverings of an algebraic curve, as the degree d is smaller than $2g+1$. Thus, investigating the normal generation of line bundles on multiple coverings can be an effective approach to the question of normal generation of line bundles on curves. In this paper, we obtain sufficient conditions for line bundles on a multiple covering to be normally generated and also obtain sufficient conditions for the failure of the property of normal generation.

1. Introduction

Throughout this paper, C is a smooth irreducible algebraic curve of genus g over an algebraically closed field of characteristic 0. A line bundle $\mathcal{L}$ on C is said to be normally generated if $\mathcal{L}$ is very ample and $\varphi_{\mathcal{L}}(C)$ is projectively normal for its associated morphism $\varphi_{\mathcal{L}} : C \to \mathbb{P}H^0(C, \mathcal{L})$.

It is well known that any line bundle of degree at least $2g+1$ is normally generated and a general line bundle of degree $2g$ on a non-hyperelliptic curve is normally generated (see [6,8]). It was also shown that hyperelliptic curves have no normally generated line bundles of degree less than $2g+1$ (see [5]). Thus a natural interest is to characterize the line bundles of degree near $2g$ failing to be normally generated and the curves carrying such line bundles. In [1,7,4,3], they gave the conditions under which nonspecial very ample line bundles $\mathcal{L}$ with $\deg \mathcal{L} \geqslant 2g-5$ and special very ample line

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bundles $\mathcal{L}$ with $\deg \mathcal{L} \geqslant 2g-7$ fail to be normally generated. The previous results shows that the line bundles which fail to be normally generated appear on multiple coverings and are closely connected with line bundles on the base curves of the coverings. Moreover, both the degrees of those coverings and the genera of base curves become larger as we proceed with these works. Thus, investigating the normal generation of line bundles on multiple coverings is a natural approach to the question of normal generation of line bundles on curves.

The purpose of this work is to detect the conditions for the normal generation of line bundles on multiple coverings. For a nonspecial line bundle $\mathcal{L}$ on a multiple covering C of a smooth curve C', we introduce, in Section 3, a kind of concrete description such as

$$\mathcal{L} \sim \mathcal{K}_C - \phi^* g_t^0 - \sum_{i=1}^{b} B_i + E \qquad (*)$$

for some $B_i \geqslant 0$ with $\phi(B_i) = Q_i \in C'$, $E > 0$ on C and g_t^0 on C' satisfying $h^0(C, \phi^* g_t^0 + \sum_{i=1}^b B_i) = 1$, $\operatorname{supp}(\phi^* g_t^0 + \sum_{i=1}^b B_i) \cap \operatorname{supp}(E) = \emptyset$ and $\sum_{i=1}^b B_i \ngeqslant \phi^* Q$ for any $Q \in C'$. Here, ϕ is the covering morphism. Our results on nonspecial line bundles are described in terms of $\sum_{i=1}^b B_i$ and E. Using this description, we also construct nonspecial line bundles possessing an expected normal generation property on multiple coverings.

The results of this work are as follows. Here, we assume that C admits an n-fold covering morphism $\phi : C \to C'$ for a smooth curve C' of genus p. In the result (ii), we also denote a special line bundle by $\mathcal{L} \sim K_C - \phi^* g_d^r - \sum_{i=1}^b B_i$ and say that $\deg E = 0$ for the convenience. For the following (i) and (ii), we additionally assume that ϕ is simple with $g > np$ and define numbers

$$\mu := \left[\frac{2n(n-1)p}{g-np}\right], \qquad \delta := \min\left\{\frac{g+5-2\deg E}{6}, \frac{g-np}{n-1} - 2, \frac{2(2n+\mu-3)}{(2n+\mu-1)^2} g\right\}.$$

Here, ϕ is said to be simple if ϕ does not factor through a nontrivial morphism.

(i) Let $\mathcal{L}$ be a nonspecial line bundle on C with $\mathcal{L} \sim \mathcal{K}_C - \phi^* g_t^0 - \sum_{i=1}^b B_i + E$ as $(*)$. Assume $\deg(\phi^* g_t^0 + \sum_{i=1}^b B_i) < \delta - 1$, equivalently $\operatorname{Cliff}(\mathcal{L}) < \delta - a + 1$. Then we have the following:

For $n \geqslant 3$, $\mathcal{L}$ is normally generated if $\deg B_i \leqslant n-3$ for all $i \in \{1, \ldots, b\}$.

For $n = 2$, $\mathcal{L}$ is normally generated if $\sum_{i=1}^b B_i = 0$, $h^0(C, \phi^* g_t^0 + P + Q) = 1$ for any $P, Q \in C$.

(ii) Let $\mathcal{L}$ be a special very ample line bundle on C with $\deg \mathcal{L} > \frac{3g-3}{2}$. Assume $\mathcal{L} \sim \mathcal{K}_C - \phi^* g_d^r - \sum_{i=1}^b B_i$ for a g_d^r on C' and B_i's on C such that $\deg B_i \leqslant n-1$ and $\phi(B_i) = Q_i$ for distinct points $Q_i \in C$. Then $\mathcal{L}$ is normally generated if $b_1 \leqslant 3$ and $\deg \mathcal{L} > 2g+1-2h^1(C, \mathcal{L}) - \delta$, where $b_1 := \#\{i \mid \deg B_i = n-1,\ i = 1, \ldots, b\}$.

On the other hand, we also obtain examples of line bundles $\mathcal{L}$ which fail to be normally generated on multiple coverings if $\deg B_i = n-1$ with $h^1(C, \mathcal{L}) = 0$ and $b_1 = 4$ with $h^1(C, \mathcal{L}) \geqslant 1$, respectively. Hence the results (i) and (ii) are meaningful in some sense.

Using the result (i), for any $d > \max\{2g-2p+2, 2g-\frac{g}{6}+5\}$, we construct nonspecial normally generated line bundles of degree d on double coverings with $g \geqslant 4p$ (see Corollary 9, whose result also contains the cases $n \geqslant 3$), unless C is a double covering of a hyperelliptic curve. Note that a double covering of genus g admits no nonspecial normally generated line bundles $\mathcal{L}$ with $g+5 \leqslant \deg \mathcal{L} \leqslant 2g-3p$ (see [5]).

(iii) Let $\mathcal{L}$ be a nonspecial very ample line bundle on C such that $\mathcal{L} \sim \mathcal{K}_C - \phi^* g_t^0 - \sum_{i=1}^b (\phi^*(Q_i) - P_i) + E$, $\phi(P_i) = Q_i$, and $h^1(C, \mathcal{L}^2(-\sum_{i=1}^b P_i - E)) = 0$. Then $\mathcal{L}$ fails to be normally generated if $a \geqslant 3$ and $a + b > \frac{(a+b-r)(a+b-r-1)}{2}$, where $a = \deg E$, $r = h^0(C', g_t^0 + \sum_{i=1}^b Q_i) - 1$.

(iv) Let $\mathcal{L}$ be a special very ample line bundle on C. Assume $\mathcal{K}_C \otimes \mathcal{L}^{-1} = \phi^* \mathcal{N}(-\sum_{i=1}^b P_i)$ for some line bundle $\mathcal{N}$ on C' and $\sum_{i=1}^b P_i$ on C. Set $c := h^0(C, \phi^* \mathcal{N}) - h^0(C, \phi^* \mathcal{N}(-\sum_{i=1}^b P_i))$. Then $\mathcal{L}$ fails to be normally generated if $b > \frac{(b-c)(b-c+1)}{2}$ and $h^1(C, \mathcal{L}^2(-\sum_{i=1}^b P_i)) = 0$.

In this paper, g^r_d means a linear series of dimension r and degree d. In particular, g^0_d also denotes the corresponding effective divisor of degree d. The notation $\mathcal{L} - g^r_d$ means $\mathcal{L}(-D)$ for $D \in g^r_d$ and a line bundle $\mathcal{L}$. For a divisor D and a line bundle $\mathcal{L}$ on a smooth curve C, we also denote $h^i(C, \mathcal{O}_C(D))$ by $h^i(C, D)$ and $\mathcal{O}_C(D) \subseteq \mathcal{L}$ by $D \leqslant \mathcal{L}$. And $\mathcal{K}_C$ means the canonical line bundle on C. The Clifford index of a smooth curve C is defined by $\operatorname{Cliff}(C) := \min\{\operatorname{Cliff}(\mathcal{L})\colon\ h^0(C,\mathcal{L}) \geqslant 2,\ h^1(C,\mathcal{L}) \geqslant 2\}$, where $\operatorname{Cliff}(\mathcal{L}) = \deg \mathcal{L} - 2h^0(C,\mathcal{L}) + 2$. For a base point free line bundle $\mathcal{L}$, we denote by $\varphi_{\mathcal{L}}$ the morphism associated to $\mathcal{L}$.

2. Preliminaries

Before going into main theorems, we consider some lemmas which will be used in our study.

Lemma 1. *Let $\mathcal{L}$ be a very ample line bundle on a smooth curve C. Consider the embedding $C \subset \mathbb{P}H^0(C,\mathcal{L}) = \mathbb{P}^r$ defined by $\mathcal{L}$. Then $\mathcal{L}$ fails to be normally generated if there exists an effective divisor D on C such that* $\deg D > \frac{(n+1)(n+2)}{2}$ *and $H^1(C, \mathcal{L}^2(-D)) = 0$, where $n := \dim \overline{D}$, $\overline{D} := \bigcap\{H \in H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \mid H, C \geqslant D\}$.*

Proof. Set

$$\Psi := \bigl\{S \in H^0\bigl(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2)\bigr) \bigm| S\colon \text{ quadric cone with vertex } \overline{D}\bigr\}.$$

Then, $\Psi \subseteq H^0(\mathbb{P}^r, \mathcal{I}_{D/\mathbb{P}^r}(2))$ and

$$\dim \Psi = \dim \operatorname{Grass}(r-n-1, r) + h^0\bigl(\mathbb{P}^{r-n-1}, \mathcal{O}_{\mathbb{P}^{r-n-1}}(2)\bigr) = \frac{r^2+3r-n^2-3n}{2}.$$

This yields

$$h^0\bigl(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2)\bigr) - h^0\bigl(\mathbb{P}^r, \mathcal{I}_{D/\mathbb{P}^r}(2)\bigr) \leqslant h^0\bigl(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2)\bigr) - \dim \Psi < \deg D$$

for $\deg D > \frac{(n+1)(n+2)}{2}$. From this, we get $h^1(\mathbb{P}^r, \mathcal{I}_{D/\mathbb{P}^r}(2)) \neq 0$ by the exact sequence $0 \to \mathcal{I}_{D/\mathbb{P}^r}(2) \to \mathcal{O}_{\mathbb{P}^r}(2) \to \mathcal{O}_D(2) \to 0$. Combining this with the exact sequence $0 \to \mathcal{I}_{C/\mathbb{P}^r}(2) \to \mathcal{I}_{D/\mathbb{P}^r}(2) \to \mathcal{I}_{D/C}(2) \to 0$, we obtain $h^1(\mathbb{P}^r, \mathcal{I}_{C/\mathbb{P}^r}(2)) \neq 0$, since $h^1(\mathbb{P}^r, \mathcal{I}_{D/C}(2)) = h^1(C, \mathcal{L}^2(-D)) = 0$. Thus the result follows. $\square$

This lemma is practical in verifying the failure of normal generation of line bundles on curves, since its conditions are purely numerical and hence can be computed by theories about linear series on curves. On the other hand, we have the following lemma from the proof of Theorem 3 in [1], which is useful to demonstrate the property of normal generation. In fact, it provides another line bundle with higher speciality in case a given line bundle fails to be normally generated.

Lemma 2. *(See Green and Lazarsfeld [1].) Let $\mathcal{L}$ be a very ample line bundle on C with* $\deg \mathcal{L} > \frac{3g-3}{2} + \epsilon$, *where $\epsilon = 0$ if $\mathcal{L}$ is special, $\epsilon = 2$ if $\mathcal{L}$ is nonspecial. If $\mathcal{L}$ fails to be normally generated, then there exists a line bundle $\mathcal{A} \simeq \mathcal{L}(-R)$, $R > 0$, such that*:

(i) $\operatorname{Cliff}(\mathcal{A}) \leqslant \operatorname{Cliff}(\mathcal{L})$,
(ii) $\deg \mathcal{A} \geqslant \frac{g-1}{2}$,
(iii) $h^0(C,\mathcal{A}) \geqslant 2$ *and* $h^1(C,\mathcal{A}) \geqslant h^1(C,\mathcal{L}) + 2$.

Since Lemma 2 plays an important role in this work, we frequently compute the Clifford indices of line bundles.

Lemma 3. *Let $\mathcal{M}$ be a base point free line bundle on C with $\deg \mathcal{M} \leqslant 2g-2$ such that its associated morphism $\varphi_{\mathcal{M}}$ is birational.*

(i) *If* $\deg \mathcal{M} \geqslant g-1$, *then* $\mathrm{Cliff}(\mathcal{M}) \geqslant \frac{\deg \mathcal{K}_C \otimes \mathcal{M}^{-1}}{3}$.
(ii) *If* $\deg \mathcal{M} \leqslant g-1$, *then*

$$\mathrm{Cliff}(\mathcal{M}) \geqslant \frac{g}{3} - 1 \quad \text{for } l = 2,$$

$$\mathrm{Cliff}(\mathcal{M}) > \frac{2(l-1)}{(l+1)^2} g - 1 \quad \text{for } l \geqslant 3,$$

where $l := [\frac{2g}{\deg \mathcal{M} - 1}]$.

Proof. Set $\alpha := h^0(C, \mathcal{M}) - 1$ and $d := \deg \mathcal{M}$. First, assume $d \geqslant g-1$. Then $\alpha \leqslant \frac{2d-g+1}{3}$ by Castelnuovo's genus bound, and hence $\mathrm{Cliff}(\mathcal{M}) = d - 2\alpha \geqslant \frac{\deg \mathcal{K}_C \otimes \mathcal{M}^{-1}}{3}$. Next, suppose $d \leqslant g-1$. Set $l = [\frac{2g}{d-1}]$. Then $\frac{2g}{l} + 1 \geqslant d > \frac{2g}{l+1} + 1$. If $l = 2$, then Castelnuovo's genus bound yields $\alpha \leqslant \frac{3d-g+3}{6}$ (see Lemma 8 in [2]), which implies $\mathrm{Cliff}(\mathcal{M}) = d - 2\alpha \geqslant \frac{g-3}{3}$. If $l \geqslant 3$, then by Lemma 9 in [2] we have $\alpha \leqslant \frac{d+l}{l+1}$, and so

$$\mathrm{Cliff}(\mathcal{M}) \geqslant d - \frac{2d+2l}{l+1} = \frac{(l-1)d - 2l}{l+1} > \frac{2(l-1)}{(l+1)^2} g - 1$$

for $d > \frac{2g}{l+1} + 1$. Thus the result follows. □

To prove our main results, we will use a figure which draws the correspondence between points of C and C' for a multiple covering morphism $\phi : C \to C'$. By such a figure, some computations about line bundles will be simplified if those line bundles are composed with ϕ. To do such a work, we need the following.

Lemma 4. *Assume that C admits a simple n-fold covering morphism $\phi : C \to C'$ for a smooth curve C' of genus p. And let $\mathcal{M}$ be a line bundle on C with $h^0(C, \mathcal{M}) \geqslant 3$ and $\mathrm{Cliff}(\mathcal{M}) < \frac{g-np}{n-1} - 3$. Then $\mathcal{M}(-B)$ is either simple or composed with ϕ, where B is the base locus of $\mathcal{M}$.*

Proof. The condition $\mathrm{Cliff}(\mathcal{M}) < \frac{g-np}{n-1} - 3$ also implies $\mathrm{Cliff}(\mathcal{M}(-B)) < \frac{g-np}{n-1} - 3$. Thus we may assume $\mathcal{M}$ is generated by its global sections. Suppose $\mathcal{M}$ is neither simple nor composed with ϕ. Set $d := \deg \mathcal{M}$, $\alpha := h^0(C, \mathcal{M}) - 1$ and $m := \deg \varphi_{\mathcal{M}}$.

Consider a birational projection $\pi : \varphi_{\mathcal{M}}(C) \to \mathbb{P}^2$ from general $(\alpha-2)$-points $\sum_{i=1}^{\alpha-2} Q_i$ of $\varphi_{\mathcal{M}}(C)$. Then the morphism $\pi \circ \varphi_{\mathcal{M}} : C \to \mathbb{P}^2$ is associated to the line bundle $\mathcal{M}(-\sum_{i=1}^{\alpha-2} \varphi_{\mathcal{M}}^*(Q_i))$. Thus we have the following commutative diagram.

$$\begin{tikzcd} C \arrow[r, "\varphi_{\mathcal{M}}"] \arrow[dr, "\varphi_{\mathcal{M}(-\sum_{i=1}^{\alpha-2} \varphi_{\mathcal{M}}^*(Q_i))}"'] & \varphi_{\mathcal{M}}(C) \subset \mathbb{P}^{\alpha} \arrow[d, "\pi"] \\ & \mathbb{P}^2 \end{tikzcd}$$

If $\varphi_{\mathcal{M}(-\sum_{i=1}^{\alpha-2}\varphi_{\mathcal{M}}^*(Q_i))}$ is composed with ϕ, then $\varphi_{\mathcal{M}(-\sum_{i=1}^{\alpha-2}\varphi_{\mathcal{M}}^*(Q_i))} = \mu \circ \phi$ for a morphism μ of degree $\geqslant 2$. Then there is a rational morphism $\nu : C' \to \varphi_{\mathcal{M}}(C)$ such that the following diagram commutes as rational morphisms, since π is birational.

$$\begin{tikzcd} C \arrow[r, "\varphi_{\mathcal{M}}"] \arrow[dr, "\phi"'] & \varphi_{\mathcal{M}}(C) \arrow[r, "\pi"] & \varphi_{\mathcal{M}(-\sum_{i=1}^{\alpha-2}\varphi_{\mathcal{M}}^*(Q_i))}(C) \\ & C' \arrow[u, "\nu"] \arrow[ur, "\mu"'] & \end{tikzcd}$$

The smoothness of C' implies that the rational map ν is regular, which contradicts that $\varphi_{\mathcal{M}}$ is not composed with ϕ. Accordingly, the morphism $\varphi_{\mathcal{M}(-\sum_{i=1}^{\alpha-2}\varphi_{\mathcal{M}}^*(Q_i))}$ is not composed with ϕ, whence for a general subseries $g^1_{d-m\alpha+2m}$ of $|\mathcal{M}(-\sum_{i=1}^{\alpha-2}\varphi_{\mathcal{M}}^*(Q_i))|$ the product morphism $\phi \times \varphi_{g^1_{d-m\alpha+2m}}$ is birational since ϕ is simple. Applying the Castelnuovo–Severi inequality, we obtain $g \leqslant (n-1)(d - m\alpha + 2m - 1) + np$ and hence

$$\mathrm{Cliff}(\mathcal{M}) = d - 2\alpha \geqslant \frac{g-np}{n-1} - 2m + 1 + (m-2)\alpha \geqslant \frac{g-np}{n-1} - 3$$

for $\alpha \geqslant 2$, $m \geqslant 2$. It is a contradiction to $\mathrm{Cliff}(\mathcal{M}) < \frac{g-np}{n-1} - 3$. Thus the result follows. □

Lemma 5. *Assume that C admits a simple n-fold covering morphism $\phi : C \to C'$ for a smooth curve C' of genus p with $g > np$. Let $\mathcal{M}$ be a line bundle on C with $h^0(C, \mathcal{M}) \geqslant 2$. Then $\mathcal{M}(-B)$ is composed with the morphism ϕ if*

$$\mathrm{Cliff}(\mathcal{M}) < \min\left\{\frac{g-np}{n-1} - 3, \frac{2(2n+\mu-3)}{(2n+\mu-1)^2}g - 1, \frac{\deg \mathcal{K}_C \otimes \mathcal{M}^{-1}}{3}\right\},$$

where $\mu := [\frac{2n(n-1)p}{g-np}]$ and B is the base locus of $\mathcal{M}$.

Proof. We may assume that the line bundle $\mathcal{M}$ is base point free, since $\mathrm{Cliff}(\mathcal{M}(-B)) \leqslant \mathrm{Cliff}(\mathcal{M})$. Set $d := \deg \mathcal{M}$, $\alpha := h^0(C, \mathcal{M}) - 1$.

Assume $\alpha = 1$ and $\mathcal{M}$ is not composed with ϕ. According to the Castelnuovo–Severi inequality we obtain $g \leqslant (n-1)(d-1) + np$, since ϕ is simple. Then

$$\frac{g-np}{n-1} - 1 \leqslant d - 2 = \mathrm{Cliff}(\mathcal{M}) < \frac{g-np}{n-1} - 3,$$

which cannot occur. Hence $\varphi_{\mathcal{M}}$ is composed with the covering morphism ϕ.

Consider the other cases $\alpha \geqslant 2$. Assume $\varphi_{\mathcal{M}}$ is birational. Then by the simpleness of the covering morphism ϕ and the Castelnuovo–Severi inequality, we have $g \leqslant (n-1)(d-1) + np$ and so $(\frac{g-np}{g})\frac{2g}{d-1} \leqslant 2(n-1)$. Since $g > np$,

$$\left[\frac{2g}{d-1}\right] \leqslant \left[2(n-1)\left(1 + \frac{np}{g-np}\right)\right] \leqslant 2(n-1) + \mu,$$

where $\mu := [\frac{2n(n-1)p}{g-np}]$. Then from Lemma 3 we obtain

$$\mathrm{Cliff}(\mathcal{M}) \geqslant \min\left\{\frac{2(2n+\mu-3)}{(2n+\mu-1)^2}g - 1, \frac{\deg \mathcal{K}_C \otimes \mathcal{M}^{-1}}{3}\right\},$$

since $\frac{g}{3} - 1 \geqslant \frac{2(l-1)}{(l+1)^2} g - 1$ for $l \geqslant 2$. It cannot happen by the assumptions for $\mathrm{Cliff}(\mathcal{M})$. Thus we have $m \geqslant 2$, and hence $\mathcal{M}(-B)$ is composed with the covering morphism ϕ by Lemma 4 and the condition $\mathrm{Cliff}(\mathcal{L}) < \frac{g-np}{n-1} - 3$. □

3. Normal generation of nonspecial line bundles on multiple coverings

In this section, we investigate the normal generation of nonspecial line bundles on multiple coverings. To do this, we consider a concrete description for nonspecial line bundles on a smooth curve. Let $\mathcal{L}$ be a nonspecial line bundle on a smooth curve C. There exists a divisor $E > 0$ such that

$$h^0\big(C, \mathcal{K}_C \otimes \mathcal{L}^{-1}(E)\big) = 1, \qquad h^0\big(C, \mathcal{K}_C \otimes \mathcal{L}^{-1}\big(E'\big)\big) = 0$$

for $E' < E$. Then we have

$$\mathcal{L} \sim \mathcal{K}_C - g^0_d + E$$

for a g^0_d on C satisfying $\mathrm{supp}(g^0_d) \cap \mathrm{supp}(E) = \emptyset$, where g^0_d means a degree d divisor with $h^0(C, g^0_d) = 1$. Note that we have $\deg \mathcal{K}_C \otimes \mathcal{L}^{-1}(E) \leqslant g$ since $h^0(C, \mathcal{K}_C \otimes \mathcal{L}^{-1}(E)) = 1$.

Lemma 6. *Let $\mathcal{L}$ be a nonspecial line bundle such that $\mathcal{L} \sim \mathcal{K}_C - g^0_d + E$ for a divisor $E > 0$ and a g^0_d on C satisfying $\deg E \geqslant 3$ and $\mathrm{supp}(g^0_d) \cap \mathrm{supp}(E) = \emptyset$. Then $\mathcal{L}$ is very ample if $h^0(C, g^0_d + P + Q) = 1$ for any $P, Q \in C$.*

Proof. If $\mathcal{L}$ is not very ample, there are $P, Q \in C$ such that $h^0(C, \mathcal{K}_C \otimes \mathcal{L}^{-1}(P + Q)) = 1$, which implies that there is an effective divisor D on C,

$$g^0_d - E + P + Q \sim D \quad \text{and so} \quad g^0_d + P + Q \sim E + D.$$

Then we have $h^0(C, g^0_d + P + Q) \geqslant 2$, since $\deg E \geqslant 3$ and $\mathrm{supp}(g^0_d) \cap \mathrm{supp}(E) = \emptyset$. Thus the result holds. □

Suppose C admits a multiple covering morphism $\phi : C \to C'$ for some smooth curve C'. Then, $g^0_d = \phi^* g^0_t + B$ for g^0_t on C' and $B \geqslant 0$ on C such that $B \ngeqslant \phi^* Q$ for any $Q \in C'$. Thus a nonspecial line bundle $\mathcal{L}$ on the multiple covering C can be written by

$$\mathcal{L} \sim \mathcal{K}_C - \phi^* g^0_t - \sum_{i=1}^{b} B_i + E$$

for some $B_i \geqslant 0$, $E > 0$ on C and g^0_t on C' such that:

(1) $h^0(C, \phi^* g^0_t + \sum_{i=1}^{b} B_i) = 1$,
(2) $\phi(B_i) = Q_i$ for $Q_i \in C'$, $\deg B_i \leqslant n - 1$,
(3) $\mathrm{supp}(\phi^* g^0_t + \sum_{i=1}^{b} B_i) \cap \mathrm{supp}(E) = \emptyset$,
(4) $\sum_{i=1}^{b} B_i \ngeqslant \phi^* Q$ for any $Q \in C'$.

Theorem 7. *Let C admit a simple n-fold covering morphism $\phi : C \to C'$ for a smooth curve C' of genus p with $g > np$. And let $\mathcal{L} \sim \mathcal{K}_C - \phi^* g^0_t - \sum_{i=1}^{b} B_i + E$ for some $B_i \geqslant 0$, $E > 0$ on C and g^0_t on C' satisfying $a := \deg E \geqslant 3$, $\phi(B_i) \neq \phi(B_j)$ for $i \neq j$, and the conditions (1)–(4) in the above. Assume $\deg(\phi^* g^0_t + \sum_{i=1}^{b} B_i) <$*

$\delta - 1$, which is equivalent to $\operatorname{Cliff}(\mathcal{L}) < \delta - a + 1$, *where* $\delta := \min\{\frac{g+5-2a}{6}, \frac{g-np}{n-1} - 2, \frac{2(2n+\mu-3)}{(2n+\mu-1)^2} g\}$ *and* $\mu := [\frac{2n(n-1)p}{g-np}]$. *Then we have the following*:

(i) *For* $n \geqslant 3$, $\mathcal{L}$ *is normally generated if* $\deg B_i \leqslant n-3$ *for all* $i \in \{1, \ldots, b\}$.
(ii) *For* $n = 2$, $\mathcal{L}$ *is normally generated if* $\sum_{i=1}^{b} B_i = 0$, $h^0(C, \phi^* g_t^0 + P + Q) = 1$ *for any* $P, Q \in C$.

Proof. We claim that $\mathcal{L}$ is very ample. In case $n = 2$, Lemma 6 implies the very ampleness of $\mathcal{L}$ by the condition that $h^0(C, \phi^* g_t^0 + P + Q) = 1$ for any $P, Q \in C$. Assume $\mathcal{L}$ is not very ample for $n \geqslant 3$. According to Lemma 6, we have $h^0(C, \phi^* g_t^0 + \sum_{i=1}^{b} B_i + P + Q) \geqslant 2$ for some $P, Q \in C$, which gives

$$\operatorname{Cliff}\left(\phi^* g_t^0 + \sum_{i=1}^{b} B_i + P + Q\right) \leqslant \operatorname{Cliff}\left(\phi^* g_t^0 + \sum_{i=1}^{b} B_i\right) < \delta - 1,$$

for $h^0(C, \phi^* g_t^0 + \sum_{i=1}^{b} B_i) = 1$. Since $\deg(\phi^* g_t^0 + \sum_{i=1}^{b} B_i + P + Q) \leqslant g - 1$, Lemma 5 implies that the base point free part of $\mathcal{O}_C(\phi^* g_t^0 + \sum_{i=1}^{b} B_i + P + Q)$ is composed with the covering morphism ϕ, which cannot occur for $\deg B_i \leqslant n - 3$ for all $i \in \{1, \ldots, k\}$.

Suppose $\mathcal{L}$ fails to be normally generated. By Lemma 2, C has a line bundle $\mathcal{A} \sim \mathcal{L}(-R)$ with $R > 0$, satisfying the conditions in Lemma 2. There is an effective divisor D associated to the line bundle $\mathcal{K}_C \otimes \mathcal{A}^{-1}$, since $h^0(C, \mathcal{K}_C \otimes \mathcal{A}^{-1}) \geqslant 2$ by Lemma 2(iii). Then we get

$$D \sim \phi^* g_t^0 + \sum_{i=1}^{b} B_i - E + R, \quad \text{and hence} \quad D + E \sim \phi^* g_t^0 + \sum_{i=1}^{b} B_i + R,$$

which yields

$$\mathcal{K}_C \otimes \mathcal{A}^{-1}(E) \sim \mathcal{O}_C(D + E) \sim \mathcal{O}_C\left(\phi^* g_t^0 + \sum_{i=1}^{b} B_i + R\right).$$

We will prove the result by dividing two cases as follows:

Case 1. E is contained in the base locus of $\mathcal{K}_C \otimes \mathcal{A}^{-1}(E)$.

Case 2. E is not contained in the base locus of $\mathcal{K}_C \otimes \mathcal{A}^{-1}(E)$.

Proof of Case 1. The assumption of this case implies $\phi^* g_t^0 + \sum_{i=1}^{b} B_i + R \geqslant E$ as divisors, since $\mathcal{K}_C \otimes \mathcal{A}^{-1}(E) \sim \mathcal{O}_C(\phi^* g_t^0 + \sum_{i=1}^{b} B_i + R)$. Thus we get $R \geqslant E$ by the condition $\operatorname{supp}(\phi^* g_t^0 + \sum_{i=1}^{b} B_i) \cap \operatorname{supp}(E) = \emptyset$. Consequently, we have

$$\mathcal{K}_C \otimes \mathcal{A}^{-1} \sim \mathcal{O}_C\left(\phi^* g_t^0 + \sum_{i=1}^{b} B_i + (R - E)\right), \quad R - E \geqslant 0.$$

Let $\mathcal{M} := \mathcal{K}_C \otimes \mathcal{A}^{-1}(-\widetilde{B})$, where $\widetilde{B}$ is the base locus of $\mathcal{K}_C \otimes \mathcal{A}^{-1}$. The condition $\deg A \geqslant \frac{g-1}{2}$ (see Lemma 2) gives $\frac{\deg \mathcal{K}_C \otimes \mathcal{M}^{-1}}{3} \geqslant \frac{g-1}{6}$. Thus Lemma 5 implies that $\mathcal{M}$ is composed with the covering morphism ϕ, since the condition $\operatorname{Cliff}(\mathcal{A}) \leqslant \operatorname{Cliff}(\mathcal{L})$ in Lemma 2 and the assumption $\operatorname{Cliff}(\mathcal{L}) < \delta - a + 1$ yield

$$\operatorname{Cliff}(\mathcal{M}) \leqslant \operatorname{Cliff}\big(\mathcal{K}_C \otimes \mathcal{A}^{-1}\big) < \delta - 1.$$

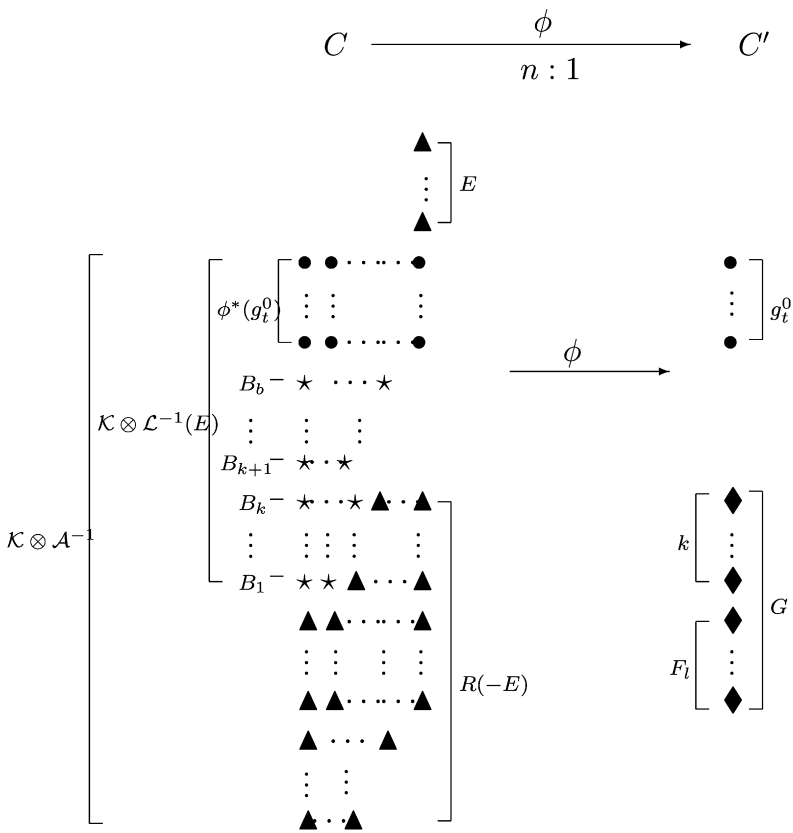

Fig. 1. $\mathcal{L}$ is nonspecial with $R \geqslant E$.

Since R contains E, we set $R(-E) = \phi^*(F_l) + R_0$ for a divisor $R_0 \geqslant 0$ on C and a degree l divisor F_l on C' such that $R_0 \ngeqslant \phi^* Q$ for any $Q \in C'$. Assume $\phi(B_1), \ldots, \phi(B_k)$ are only the members of $\sum_{i=1}^{b} B_i$ such that $\phi^*(\phi(B_i)) - B_i \leqslant R$. Set $G := F_l + \sum_{i=1}^{k} \phi(B_i)$. Then $\phi^*(g_t^0 + G)$ corresponds to the pullback part of $\mathcal{K} \otimes \mathcal{A}^{-1}$ via the covering morphism ϕ.

For a better understanding, consider Fig. 1 which draws the correspondence of points on curves C and C'. Here,

(i) E: the sum of the points being arranged as triangles on the upper left side,
(ii) $\mathcal{K}_C \otimes \mathcal{L}^{-1}(E)$: the sum of the points being arranged as black dots and stars on the left side,
(iii) $\sum_{i=1}^{b} B_i$: the sum of the points being arranged as stars on the left side,
(iv) R: the sum of the points being arranged as triangles on the left side,
(v) $\mathcal{K}_C \otimes \mathcal{A}^{-1}$: the sum of the points being arranged as black dots and stars on the left side and triangles on the lower left side,
(vi) g_t^0: the sum of the points being arranged as black dots on the right side,
(vii) G: the sum of the points being arranged as $(l+k)$-black diamonds on the right side,
(viii) F_l: the sum of the points being arranged as the assigned l-black diamonds on the right side.

From Fig. 1, we easily see that $h^0(C, \mathcal{K}_C \otimes \mathcal{A}^{-1}) \leqslant k + l + 1$, since $\varphi_{\mathcal{M}}$ is composed with ϕ. Thus

$$\deg R \leqslant 2\bigl(h^0\bigl(C, \mathcal{K}_C \otimes \mathcal{A}^{-1}\bigr) - h^0\bigl(C, \mathcal{K}_C \otimes \mathcal{L}^{-1}\bigr)\bigr) \leqslant 2(k + l + 1),$$

since $\mathrm{Cliff}(\mathcal{K}_C \otimes \mathcal{A}^{-1}) \leqslant \mathrm{Cliff}(\mathcal{K}_C \otimes \mathcal{L}^{-1})$ and $\mathcal{A} \cong \mathcal{L}(-R)$ by Lemma 2. On the other hand, we have

$$\deg R \geqslant a + nl + 3k \geqslant 3 + nl + 3k,$$

since $a := \deg E \geqslant 3$, $R(-E) \geqslant \phi^* F_l + \sum_{i=1}^{k} (\phi^*(\phi(B_i)) - B_i)$, $\deg B_i \leqslant n-3$ for $n \geqslant 3$, and $k = 0$ for $n = 2$. It cannot happen. Thus the theorem is proved for Case 1. $\square$

Proof of Case 2. Since E is not contained in the base locus of $\mathcal{K}_C \otimes \mathcal{A}^{-1}(E)$, we have $h^0(C, \mathcal{K}_C \otimes \mathcal{A}^{-1}(E)) \geqslant h^0(C, \mathcal{K}_C \otimes \mathcal{A}^{-1}) + 1$, which yields

$$\mathrm{Cliff}\big(\mathcal{K}_C \otimes \mathcal{A}^{-1}(E)\big) \leqslant \mathrm{Cliff}\big(\mathcal{K}_C \otimes \mathcal{A}^{-1}\big) + a - 2.$$

Since $\mathcal{L} \sim \mathcal{K}_C - \phi^* g_t^0 - \sum_{i=1}^{b} B_i + E$ and $h^0(C, \phi^* g_t^0 + \sum_{i=1}^{b} B_i) = 1$,

$$\mathrm{Cliff}(\mathcal{A}) \leqslant \mathrm{Cliff}(\mathcal{L}) = nt + \sum_{i=1}^{b} \deg B_i - a + 2 = \mathrm{Cliff}\left(\phi^* g_t^0 + \sum_{i=1}^{b} B_i\right) - a + 2,$$

which implies $\mathrm{Cliff}(\mathcal{K}_C \otimes \mathcal{A}^{-1}(E)) \leqslant \mathrm{Cliff}(\phi^* g_t^0 + \sum_{i=1}^{b} B_i)$. Then we have $\mathrm{Cliff}(\mathcal{O}_C(\phi^* g_t^0 + \sum_{i=1}^{b} B_i + R)) \leqslant \mathrm{Cliff}(\mathcal{O}_C(\phi^* g_t^0 + \sum_{i=1}^{b} B_i))$ by the relation $\mathcal{K}_C \otimes \mathcal{A}^{-1}(E) \sim \mathcal{O}_C(\phi^* g_t^0 + \sum_{i=1}^{b} B_i + R)$. As a result,

$$\deg R \leqslant 2\left(h^0\left(C, \phi^* g_t^0 + \sum_{i=1}^{b} B_i + R\right) - h^0\left(C, \phi^* g_t^0 + \sum_{i=1}^{b} B_i\right)\right).$$

Set $\mathcal{N} := \mathcal{K}_C \otimes \mathcal{A}^{-1}(E)(-\widetilde{B})$, $\widetilde{B}$: base locus. Then we have

$$\mathrm{Cliff}(\mathcal{N}) \leqslant \mathrm{Cliff}\big(\mathcal{K}_C \otimes \mathcal{A}^{-1}(E)\big) \leqslant \mathrm{Cliff}\big(\mathcal{K}_C \otimes \mathcal{A}^{-1}\big) + a - 2 < \delta - 1,$$

since $\mathrm{Cliff}(\mathcal{A}) \leqslant \mathrm{Cliff}(\mathcal{L}) < \delta - a + 1$. Accordingly, Lemma 5 implies that $\mathcal{N} = \mathcal{K}_C \otimes \mathcal{A}^{-1}(E)(-\widetilde{B})$ is composed with the covering morphism ϕ, since we obtain $\frac{\deg \mathcal{K}_C \otimes \mathcal{N}^{-1}}{3} \geqslant \frac{g-1-2a}{6}$ from $\mathcal{N} \leqslant \mathcal{K}_C \otimes \mathcal{A}^{-1}(E)$ and $\deg A \geqslant \frac{g-1}{2}$.

Let $R = \phi^*(F_l) + R_0$ for a divisor $R_0 \geqslant 0$ on C and a degree l divisor F_l on C' such that $R_0 \ngeqslant \phi^* Q$ for any $Q \in C'$. Assume $\phi(B_1), \dots, \phi(B_k)$ are only the members of $\sum_{i=1}^{b} B_i$ such that $\phi^*(\phi(B_i)) - B_i \leqslant R$.

Set $G := F_l + \sum_{i=1}^{k} \phi(B_i)$. Then $\phi^*(g_t^0 + G)$ corresponds to the pullback part of $\mathcal{K} \otimes \mathcal{A}^{-1}(E)$ via the covering morphism ϕ. If $n = 2$, then

$$h^0\big(C, \mathcal{K} \otimes \mathcal{A}^{-1}(E)\big) = h^0\big(C, \phi^*\big(g_t^0 + F_l\big)\big) \leqslant l,$$

since $\sum_{i=1}^{b} B_i = 0$ and $h^0(C, \phi^* g_t^0 + P + Q) = 1$ for any $P, Q \in C$. Hence,

$$2l \leqslant \deg R \leqslant 2(l-1),$$

which cannot occur. Thus we assume $n \geqslant 3$. Then we have

$$3k + nl \leqslant \deg R \leqslant 2(k + l),$$

since $\deg B_i \leqslant n-3$ for all $i \in \{1, \dots, b\}$. It cannot occur since $k + l \geqslant 1$. Thus $\mathcal{L}$ is normally generated.

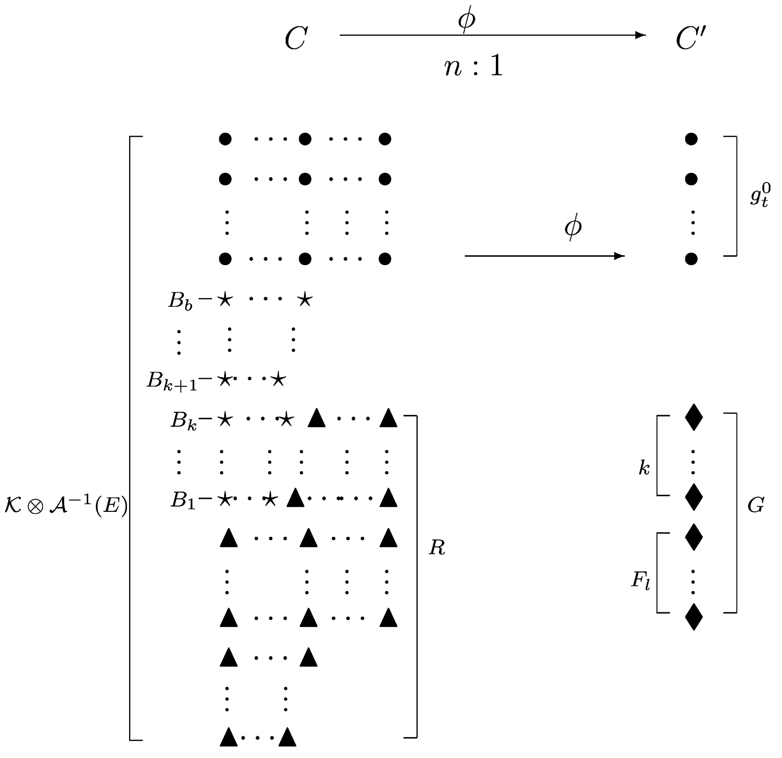

Fig. 2. $\mathcal{L}$ is nonspecial: general case.

For a better understanding, we gives Fig. 2 which draws the correspondence of points on curves C and C'. Here,

(i) $\mathcal{K}_C \otimes \mathcal{A}^{-1}(E)$: the sum of the points being arranged as black dots and stars on the left side,
(ii) $\sum_{i=1}^{b} B_i$: the sum of the points being arranged as stars on the left side,
(iii) R: the sum of the points being arranged as triangles on the left side,
(iv) g_t^0: the sum of the points being arranged as black dots on the right side,
(v) G: the sum of the points being arranged as $(l+k)$-black diamonds on the right side,
(vi) F_l: the sum of the points being arranged as the assigned l-black diamonds on the right side. $\square$

Specifically, the theorem can be further simplified for double coverings.

Corollary 8. *Let C admit a double covering morphism $\phi: C \to C'$ for a smooth curve C' of genus p with* $3g > 8(p+1)$. *Choose a g_t^0 on C' and a divisor $E > 0$ with* $\deg E \geqslant 3$ *such that* $\operatorname{supp}(\phi^* g_t^0) \cap \operatorname{supp}(E) = \emptyset$ *and $h^0(C, \phi^* g_t^0 + P + Q) = 1$ for any $P, Q \in C$. Then $\mathcal{L} \sim \mathcal{K}_C - \phi^* g_t^0 + E$ is normally generated if* $6t + \deg E < \frac{g-1}{2}$.

Proof. Since $n = 2$ and $3g > 8(p+1)$, we obtain $\frac{g-np}{n-1} - 2 \geqslant \frac{g}{6}$ and $\mu := [\frac{2n(n-1)p}{g-np}] \leqslant 6$, whence $\frac{2(2n+\mu-3)}{(2n+\mu-1)^2} > \frac{1}{6}$. Thus Theorem 7 implies that $\mathcal{L}$ is normally generated for $2t < \frac{g-1-2\deg E}{6}$, from which the result follows. $\square$

Note that Lange and Martens [5] have shown that if C is a double covering of a smooth curve C' of genus p then C admits no nonspecial normally generated line bundles $\mathcal{L}$ with $g+5 \leqslant \deg \mathcal{L} \leqslant$

$2g-3p$. On the other hand, a double covering with $g \geqslant 4p$ which is not a double covering of a hyperelliptic curve has nonspecial normally generated line bundles of degree d for any $d > \max\{2g-2p+2, 2g-\frac{g}{6}+5\}$ by the following.

Corollary 9. *Assume that C admits a simple n-fold covering morphism $\phi : C \to C'$ for a smooth curve C' of genus p with $g \geqslant \max\{n^2 p, \frac{(2n+1)^2}{3}\}$. Then C has a nonspecial normally generated line bundle of degree d for any*

$$d > \max\left\{2g-np+n, 2g-\frac{g-2n-4}{6}+n+1, 2g-\frac{2(2n-1)}{(2n+1)^2}g+n+1\right\},$$

unless C is a double covering of a hyperelliptic curve.

Proof. Choose a general effective divisor D_t of degree $t \leqslant p-1$ on C', whence D_t is a g_t^0. Choose a general divisor E on C such that $3 \leqslant \deg E \leqslant n+2$ and $\operatorname{supp}(\phi^* D_t) \cap \operatorname{supp}(E) = \emptyset$. Let $\mathcal{L} := \mathcal{K}_C - \phi^* D_t + E$.

Assume $\mathcal{L}$ is not very ample. Due to Lemma 6, we have $h^0(C, \phi^* D_t + P + Q) \geqslant 2$ for some $P, Q \in C$. Thus Lemma 5 implies that the base point free part of $\mathcal{O}_C(\phi^* D_t + P + Q)$ is composed with the covering morphism ϕ, which implies that $n = 2$ and $|\phi^* D_t + P + Q|$ is a complete $\phi^* g_{t+1}^1$. Accordingly, since D_t is a general divisor of degree t on C', we have $\dim W_{t+1}^1(C') \geqslant t-1$ for $t \leqslant p-1$, which means that C is a double covering of a hyperelliptic curve. It cannot occur. Thus $\mathcal{L}$ is very ample.

By varying the numbers t and $\deg E$ within $0 \leqslant t \leqslant p-1$ and $3 \leqslant \deg E \leqslant n+2$, we see that $\deg \mathcal{L} = 2g - nt + \deg E - 2$ can take any number $d > 2g - np + n$.

Set $a := \deg E$. From $g \geqslant n^2 p$ we get $\mu := [\frac{2n(n-1)p}{g-np}] \leqslant 2$ and so $\frac{2(2n+\mu-3)}{(2n+\mu-1)^2} g \leqslant \frac{2(2n-1)}{(2n+1)^2} g$. And the condition $g \geqslant n^2 p$ yields $\frac{g-np}{n-1} \geqslant \frac{g}{n}$, and so $\frac{g-np}{n-1} - 2 \geqslant \frac{2(2n-1)}{(2n+1)^2} g$ for $g \geqslant \frac{(2n+1)^2}{3}$. Note that $\frac{g+5-2a}{6} \geqslant \frac{g-2n+1}{6}$, for $\deg E \leqslant n+2$. According to Theorem 7, $\mathcal{L}$ is normally generated if $nt < \min\{\frac{g-2n+1}{6}-1, \frac{2(2n-1)}{(2n+1)^2} g - 1\}$. Since $\deg \mathcal{L} = 2g - 2 - nt + \deg E$ and $\deg E \leqslant n+2$, the assumption

$$\deg \mathcal{L} > 2g - \min\left\{\frac{g-2n+1}{6}-1, \frac{2(2n-1)}{(2n+1)^2} g - 1\right\} + n$$

implies $nt < \min\{\frac{g-2n+1}{6}-1, \frac{2(2n-1)}{(2n+1)^2} g - 1\}$. Consequently, the curve C admits a nonspecial normally generated line bundle of degree d for any number $d > \max\{2g-np+n, 2g-\frac{g-2n-4}{6}+n+1, 2g-\frac{2(2n-1)}{(2n+1)^2} g+n+1\}$. $\square$

In Theorem 7, we prove the normal generation of nonspecial line bundles if any B_i has degree at least $(n-3)$ for $n \geqslant 3$ and has degree $(n-2)$ (equivalently $B_i = 0$) for $n = 2$, respectively. On the other hand, we have a nonspecial line bundle which fails to be normally generated in case $\deg B_i = n-1$ as follows.

Example 10. Assume that C admits a simple n-fold covering morphism $\phi : C \to C'$ for a smooth non-rational curve C'. Take a base point free complete g_d^1 on C'. Set $g_{d-1}^0 := g_d^1 - Q$ for a point $Q \in C'$ and $B := \phi^*(Q) - P$ for $P \in \phi^* Q$. Choose a general divisor $E > 0$ on C satisfying $\deg E = 3$ and $\operatorname{supp}(B + \phi^* g_{d-1}^0) \cap \operatorname{supp}(E) = \emptyset$. Then $\mathcal{L} := \mathcal{K}_C - \phi^* g_{d-1}^0 - B + E$ is very ample and fails to be normally generated if $g > (n-1)(nd+1) + ng(C')$.

Proof. Assume $\mathcal{L}$ is not very ample. By Lemma 6, we have $D + E \sim \phi^* g_{d-1}^0 + B + P_1 + P_2$ and $h^0(C, \phi^* g_{d-1}^0 + B + P_1 + P_2) \geqslant 2$, for some $P_1, P_2 \in C$. Due to the Castelnuovo–Severi inequality,

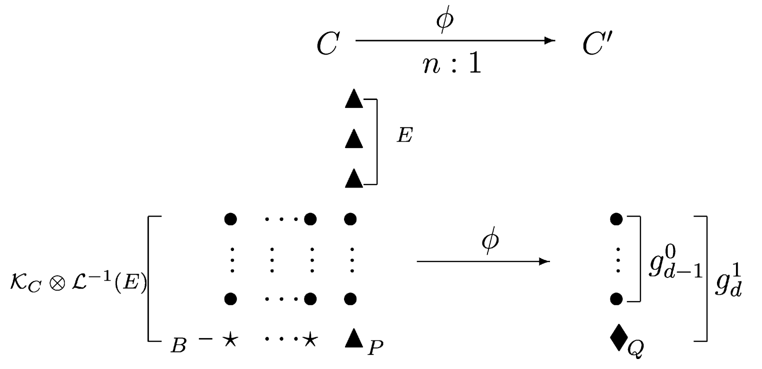

Fig. 3. Example of nonspecial line bundle.

$\mathcal{O}_C(\phi^* g^0_{d-1} + B + P_1 + P_2)$ is composed with the covering morphism ϕ, which means $|\phi^* g^0_{d-1} + B + P_1 + P_2| = \phi^* g^1_d + R$, $R \in \{P_1, P_2\}$. Note that this g^1_d is the pencil given in the above. Here, we can vary only two points P_1 and P_2 of C. It cannot occur, since by the general choice of E there are 3-dimensional family of divisors E such that $D + E \sim \phi^* g^0_{d-1} + B + P_1 + P_2$. It cannot occur. Thus $\mathcal{L}$ is very ample.

For a better understanding, we draw the correspondence of points on curves C and C' in Fig. 3. Here,

(i) E: the sum of the points being arranged as triangles on the upper left side,
(ii) $\mathcal{K}_C \otimes \mathcal{L}^{-1}(E)$: the sum of the points being arranged as black dots and stars on the left side,
(iii) B: the sum of the points being arranged as stars on the left bottom line,
(iv) g^1_d: the sum of the points being arranged as black dots and black diamonds on the right side,
(v) g^0_{d-1}: the sum of the points being arranged as black dots on the right side.

Set $D_4 := E + P$. We see that $h^0(C, \mathcal{K}_C \otimes \mathcal{L}^{-1}(D_4)) \geqslant 2$, which yields $\dim \overline{\phi_{\mathcal{L}}(D_4)} \leqslant 1$. Thus $\mathcal{L}$ fails to be normally generated by Lemma 1, since $h^1(C, \mathcal{L}^2(-D_4)) = 0$ for $\deg \mathcal{L} \geqslant g + 2$. $\square$

Using Lemma 1, we obtain a sufficient condition for the failure of normal generation of nonspecial line bundles on a multiple covering.

Theorem 11. *Assume that C admits a multiple covering morphism $\phi : C \to C'$ for a smooth curve C'. And let $\mathcal{L}$ be a very ample nonspecial line bundle. We may set $\mathcal{L} \sim \mathcal{K}_C - \phi^* g^0_t - \sum_{i=1}^b B_i + E$ as in Theorem 7. Assume $B_i = \phi^*(Q_i) - P_i$, $\phi(P_i) = Q_i$. Then $\mathcal{L}$ fails to be normally generated if $a \geqslant 3$, $a + b > \frac{(a+b-r)(a+b-r-1)}{2}$ and $h^1(C, \mathcal{L}^2(-E - \sum_{i=1}^b P_i)) = 0$, where $a := \deg E$, $r := h^0(C', g^0_t + \sum_{i=1}^b Q_i) - 1$.*

Proof. Let $D := E + \sum_{i=1}^b P_i$. Since $h^0(C, \mathcal{K}_C \otimes \mathcal{L}^{-1}(D)) \geqslant h^0(C', g^0_t + \sum_{i=1}^b Q_i)$, the Riemann–Roch Theorem gives $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-D)) \leqslant a + b - r - 1$ and so $\dim \overline{\phi_{\mathcal{L}}(D)} \leqslant a + b - r - 2$. Accordingly, $\mathcal{L}$ fails to be normally generated by Lemma 1, if $\deg D = a + b > \frac{(a+b-r)(a+b-r-1)}{2}$ and $h^1(C, \mathcal{L}^2(-E - \sum_{i=1}^b P_i)) = 0$. $\square$

4. Normal generation of special line bundles on multiple coverings

In this section, we investigate the normal generation of special line bundles on a multiple covering. Firstly, we give a sufficient condition for a special line bundle to be normally generated.

Theorem 12. *Assume that C admits a simple n-fold covering morphism $\phi : C \to C'$ for a smooth curve C' of genus p with $g > np$. Let $\mathcal{L}$ be a special very ample line bundle on C with $\deg \mathcal{L} > \frac{3g-3}{2}$. Assume $\mathcal{L} \sim \mathcal{K}_C - \phi^* g^r_d - \sum_{i=1}^b B_i$ for a linear series g^r_d on C' and effective divisors B_i on C such that $\deg B_i \leqslant n-1$, $\phi(B_i) = Q_i$ for distinct points $Q_i \in C$. Then $\mathcal{L}$ is normally generated if $b_1 \leqslant 3$ and $\deg \mathcal{L} > 2g+1-2h^1(C,\mathcal{L}) - \delta$, which is equivalent to $\mathrm{Cliff}(\mathcal{L}) < \delta - 1$. Here $b_1 := \#\{i \mid \deg B_i = n-1,\ i = 1, \ldots, b\}$, $\mu := [\frac{2n(n-1)p}{g-np}]$, and $\delta := \min\{\frac{g+5}{6}, \frac{g-np}{n-1} - 2, \frac{2(2n+\mu-3)}{(2n+\mu-1)^2} g\}$.*

Proof. Suppose $\mathcal{L}$ fails to be normally generated. Then C has a line bundle $\mathcal{A} \simeq \mathcal{L}(-R)$ with $R > 0$, satisfying the conditions in Lemma 2. Let $\mathcal{M} := \mathcal{K}_C \otimes \mathcal{A}^{-1}(-\widetilde{B})$, where $\widetilde{B}$ is the base locus of $\mathcal{K}_C \otimes \mathcal{A}^{-1}$. Note that

$$\frac{\deg \mathcal{K}_C \otimes \mathcal{M}^{-1}}{3} \geqslant \frac{\deg \mathcal{A}}{3} \geqslant \frac{g-1}{6},$$

since $\deg \mathcal{A} \geqslant \frac{g-1}{2}$ (see Lemma 2(ii)). Thus Lemma 5 implies that $\mathcal{M}$ is composed with the covering morphism ϕ, since

$$\mathrm{Cliff}(\mathcal{M}) \leqslant \mathrm{Cliff}\bigl(\mathcal{K}_C \otimes \mathcal{A}^{-1}\bigr) \leqslant \mathrm{Cliff}(\mathcal{L}) < \delta - 1.$$

As a consequence, both $\mathcal{K}_C \otimes \mathcal{A}^{-1}(-\widetilde{B})$ and $\mathcal{K}_C \otimes \mathcal{L}^{-1}(-B)$ are composed with ϕ, since $\mathcal{K}_C \otimes \mathcal{A}^{-1} > \mathcal{K}_C \otimes \mathcal{L}^{-1}$ and $h^0(C, \mathcal{K}_C \otimes \mathcal{A}^{-1}) > h^0(C, \mathcal{K}_C \otimes \mathcal{L}^{-1})$, where B is the base locus of $\mathcal{K}_C \otimes \mathcal{L}^{-1}$. Thus we have $B \geqslant \sum_{i=1}^b B_i$ and the pull-back part of $\mathcal{K}_C \otimes \mathcal{A}^{-1}$ via ϕ becomes $\phi^*(g^r_d + G)$ for some divisor $G > 0$ on C', since $\mathcal{K}_C \otimes \mathcal{A}^{-1} > \mathcal{K}_C \otimes \mathcal{L}^{-1}$. Set $R = \phi^*(F_l) + R_0$ for a divisor $R_0 \geqslant 0$ on C and a degree l divisor F_l on C' such that $R_0 \ngeqslant \phi^* Q$ for any $Q \in C'$. Let $B_i, \ldots, B_k$ be only the divisors among $\sum_{i=1}^b B_i$ such that $\phi(B_i) \leqslant G$. Then we have $G = F_l + \sum_{i=1}^k Q_i$, for $\phi(B_i) = Q_i$.

For a better understanding, we draw the correspondence of points on curves C and C' in Fig. 4. Here,

(i) $\mathcal{K}_C \otimes \mathcal{L}^{-1}$: the sum of the points being arranged as black dots and stars on the left side,
(ii) $\sum_{i=1}^b B_i$: the sum of the points being arranged as stars on the left side,
(iii) R: the sum of the points being arranged as triangles on the left side,
(iv) $\mathcal{K}_C \otimes \mathcal{A}^{-1}$: the sum of the points being arranged as black dots, stars and triangles on the left side,
(v) g^r_d: the sum of the points being arranged as black dots on the right side,
(vi) F_l: the sum of the points being arranged as the assigned l-black diamonds on the right side,
(vii) G: the sum of the points being arranged as the assigned $(l+k)$-black diamonds on the right side.

Note that the condition $\mathrm{Cliff}(\mathcal{A}) \leqslant \mathrm{Cliff}(\mathcal{L})$ implies

$$\deg R \leqslant 2\bigl(h^0\bigl(C, \mathcal{K}_C \otimes \mathcal{A}^{-1}\bigr) - h^0\bigl(C, \mathcal{K}_C \otimes \mathcal{L}^{-1}\bigr)\bigr).$$

Assume $k = 0$, then $nl \leqslant \deg R \leqslant 2(h^0(C, \mathcal{K}_C \otimes \mathcal{A}^{-1}) - h^0(C, \mathcal{K}_C \otimes \mathcal{L}^{-1})) \leqslant 2l$, which yields $n = 2$. Then, according to the very ampleness of $\mathcal{L}$ we obtain $h^0(C, \mathcal{K}_C \otimes \mathcal{A}^{-1}) - h^0(C, \mathcal{K}_C \otimes \mathcal{L}^{-1}) \leqslant l-1$, which yields $2l \leqslant \deg R \leqslant 2(l-1)$. It cannot occur, and hence we have $k \geqslant 1$.

We will divide the cases depending on the number b_1. First, we consider the case $b_1 = 0$. If $\deg B_i = n-2$ for some $i \in \{1, \ldots, k\}$, then the very ampleness of L yields $h^0(C, \mathcal{K}_C \otimes \mathcal{A}^{-1}) - h^0(C, \mathcal{K}_C \otimes \mathcal{L}^{-1}) \leqslant k + l - 1$. Thus we get $nl + 2k \leqslant \deg R \leqslant 2(k+l-1)$, which cannot occur. Thus $\deg B_i \leqslant n-3$ for all $i \in \{1, \ldots, k\}$, and so $nl + 3k \leqslant \deg R \leqslant 2(k+l)$. It is a contradiction to $k \geqslant 1$. Consequently, the case $b_1 = 0$ cannot happen.

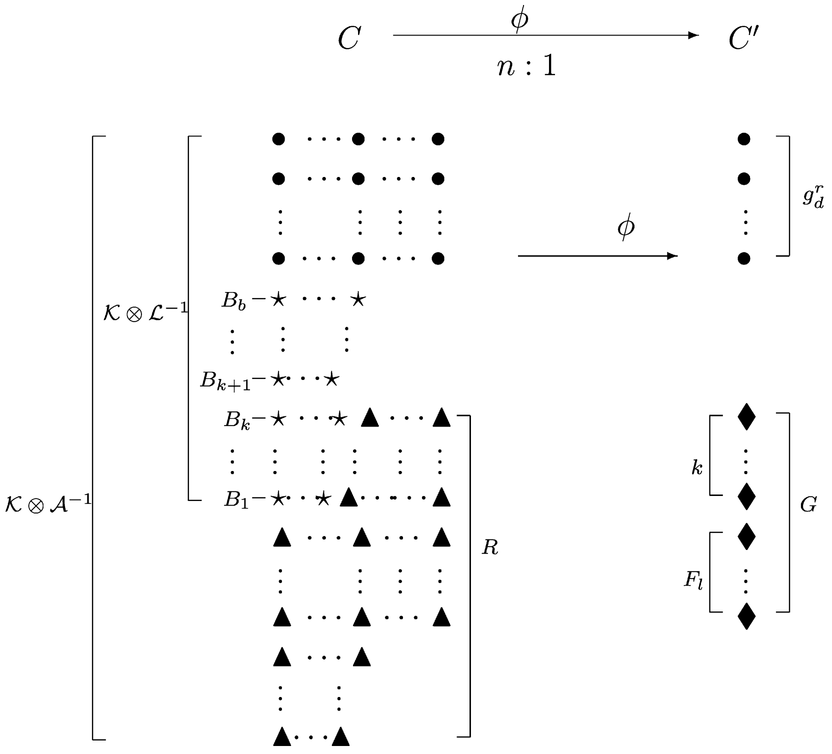

Fig. 4. $\mathcal{L}$ is special.

Next, we consider the case $b_1 = 1$. Then we have $\deg R \geqslant nl + 2k - 1$ and $h^0(C, \mathcal{K}_C \otimes \mathcal{A}^{-1}) - h^0(C, \mathcal{K}_C \otimes \mathcal{L}^{-1}) \leqslant k + l - 1$, by the base point freeness of $\mathcal{L}$. Thus we get $nl + 2k - 1 \leqslant \deg R \leqslant 2(k + l - 1)$, which is a contradiction.

As a result, we can have only the cases $b_1 \geqslant 2$, which gives $\deg R \geqslant nl + 2(k - b_1) + b_1$. Due to the very ampleness of L, the condition $b_1 \geqslant 2$ implies $h^0(C, \mathcal{K}_C \otimes \mathcal{A}^{-1}) - h^0(C, \mathcal{K}_C \otimes \mathcal{L}^{-1}) \leqslant k + l - 2$. Thus we get

$$nl + 2(k - b_1) + b_1 \leqslant \deg R \leqslant 2(k + l - 2),$$

which cannot occur for $b_1 \leqslant 3$. Thus the theorem is proved. □

Note that any n-fold covering morphism is simple in case n is prime. Specifically, if C is a double covering then we have the following.

Corollary 13. *Let C be a double covering of a smooth curve C' of genus p via a morphism ϕ with $3g > 8(p+1)$. And let $\mathcal{L}$ be a special very ample line bundle on C with $\mathcal{L} \sim K_C - \phi^* g^r_d - B$ for a g^r_d on C' and a divisor $B \geqslant 0$ on C such that $B \ngeqslant \phi^* Q$ for any $Q \in C'$. If $\deg B \leqslant 3$ and $\deg \mathcal{L} > \max\{\frac{3g-3}{2}, 2g + 1 - 2h^1(C, \mathcal{L}) - \frac{g}{6}\}$, then $\mathcal{L}$ is normally generated.*

The proof of this corollary is similar to that of Corollary 8. In fact, the condition $b_1 \leqslant 3$ of Theorem 12 is sharp in some sense, since for $b_1 = 4$ there are special very ample line bundles on multiple coverings which fail to be normally generated as follows.

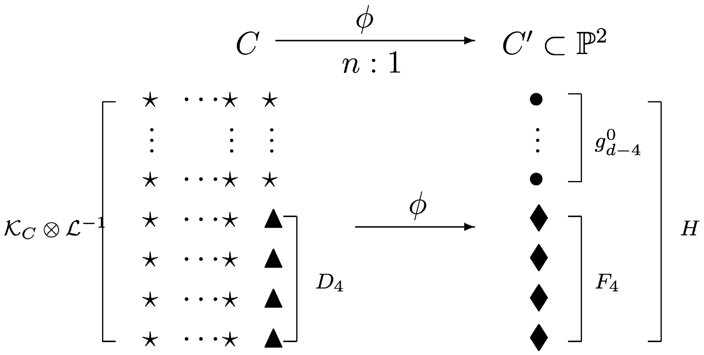

Fig. 5. Example of special line bundle.

Example 14. Let C be a simple n-fold covering of a smooth plane curve C' of degree d with $g \geqslant 3ng(C')$ and $d \geqslant n+2$. Denote the covering morphism by $\phi : C \to C'$. Let H be a general hyperplane section of C'. Choose divisors $F_4 := \sum_{i=1}^4 Q_i \leqslant H$ on C' and $D_4 := \sum_{i=1}^4 P_i$ on C such that $\phi(P_i) = Q_i$. Let $\mathcal{L}$ be the line bundle with $\mathcal{K}_C \otimes \mathcal{L}^{-1} = \phi^*(\mathcal{O}_{C'}(H))(-D_4)$. Then $\mathcal{L}$ is very ample and fails to be normally generated.

Proof. Suppose that either we have $h^1(C, \mathcal{L}) \geqslant 2$ or $\mathcal{L}$ is not very ample. Then $h^0(C, \phi^*\mathcal{O}_{C'}(H)(-D_4 + P + Q)) \geqslant 2$ for some P and Q of C. If the base point free part of $\phi^*\mathcal{O}_{C'}(H)(-D_4 + P + Q)$ is not composed with ϕ, then the Castelnuovo–Severi inequality implies $g \leqslant (n-1)(nd-3) + ng(C')$, which derives $(d-1)(d-2) \leqslant (n-1)d$ since $g \geqslant 3ng(C')$ and $g(C') = \frac{(d-1)(d-2)}{2}$. It cannot happen for $d \geqslant n+2$. Thus the base point free part of $\phi^*\mathcal{O}_{C'}(H)(-D_4 + P + Q)$ is composed with ϕ, which implies that the smooth plane curve C' of degree d has a $g^1_{a \leqslant d-2}$. It cannot occur. As a consequence, $\mathcal{L}$ is a very ample line bundle with $h^1(C, \mathcal{L}) = 1$.

For a better understanding, we draw the correspondence of points on curves C and C' in Fig. 5. Here,

(i) $\mathcal{K}_C \otimes \mathcal{L}^{-1}$: the sum of the points being arranged as stars on the left,
(ii) H: the sum of the points being arranged as black dots and black diamonds on the right,
(iii) F_4: the sum of the points being arranged as black diamonds on the right,
(iv) D_4: the sum of the points being arranged as triangles on the left.

Since $h^1(C, \mathcal{L}) = 1$ and $\mathcal{K}_C \otimes \mathcal{L}^{-1}(D_4) = \phi^*\mathcal{O}_{C'}(H)$, we have $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-D_4)) \leqslant 2$, and so $\dim \overline{\varphi_{\mathcal{L}}(D_4)} \leqslant 1$. Thus $\mathcal{L}$ fails to be normally generated due to Lemma 1, since $h^1(C, \mathcal{L}^2(-D)) = 0$. $\square$

Using Lemma 1, we have various special line bundles on multiple coverings which fail to be to be normally generated.

Theorem 15. *Assume that C admits a multiple covering morphism $\phi : C \to C'$ for a smooth curve C'. And let $\mathcal{L}$ be a special very ample line bundle on C such that $\mathcal{K}_C \otimes \mathcal{L}^{-1} = \phi^*\mathcal{N}(-\sum_{i=1}^b P_i)$ for a line bundle $\mathcal{N}$ on C' and $\sum_{i=1}^b P_i$ on C. Then $\mathcal{L}$ fails to be normally generated, if $h^1(C, \mathcal{L}^2(-\sum_{i=1}^b P_i)) = 0$ and $b > \frac{(b-c)(b-c+1)}{2}$, where $c := h^0(C, \phi^*\mathcal{N}) - h^0(C, \phi^*\mathcal{N}(-\sum_{i=1}^b P_i))$.*

Proof. Set $D_b := \sum_{i=1}^b P_i$. Then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-D_b)) = b - c$ by the Riemann–Roch Theorem, which implies $\dim \overline{\varphi_{\mathcal{L}}(D_b)} = b - c - 1$. Accordingly, the result follows from Lemma 1. $\square$

Note that, in Theorem 15, we get $b > 3$ by the very ampleness of $\mathcal{L}$ and the condition $b > \frac{(b-c)(b-c+1)}{2}$. Thus the bound $b \leqslant 3$ in Theorem 12 might be optimal.

Consider the following as an application of Theorem 15. Let C' be a linearly normal smooth curve of degree $d \geqslant 7$ in $\mathbb{P}^4$. And let C be a smooth curve of genus g admitting a simple covering morphism $\phi : C \to C'$ of degree $n \geqslant 3$ with $g > (n-1)(nd-6) + ng(C')$. Let H be a general hyperplane section of C'. Set $\mathcal{N} := \mathcal{O}_{C'}(H)$. Assume $\sum_{i=1}^{7} Q_i \leqslant H$ and $\phi(P_i) = Q_i$ for each $i = 1, \ldots, 7$. Then $\mathcal{L} := \mathcal{K}_C \otimes \phi^*\mathcal{N}^{-1}(\sum_{i=1}^{7} P_i)$ is a special very ample line bundle which fails to be normally generated. This can be shown as follows.

First, we claim $\mathcal{L}$ is very ample with $h^1(C, \mathcal{L}) = 1$. Suppose not. Then

$$h^0\left(C, \phi^*\mathcal{N}\left(-\sum_{i=1}^{7} P_i + R_1 + R_2\right)\right) \geqslant 2$$

for some $R_1 + R_2$ on C, since $h^1(C, \mathcal{L}) = h^0(C, \phi^*\mathcal{N}(-\sum_{i=1}^{7} P_i)) \geqslant 1$. According to the Castelnuovo–Severi inequality and the assumption $g > (n-1)(nd-6) + ng(C')$, the base point free part of $\phi^*\mathcal{N}(-\sum_{i=1}^{7} P_i + R_1 + R_2)$ is composed with ϕ. Let $\{Q_1, \ldots, Q_l\} := \{Q_1, \ldots, Q_7\} - \{\phi(R_1), \phi(R_2)\}$. For $n \geqslant 3$,

$$h^0\left(C', H\left(-\sum_{i=1}^{l} Q_i\right)\right) \geqslant h^0\left(C, \phi^*\mathcal{N}\left(-\sum_{i=1}^{7} P_i + R_1 + R_2\right)\right) \geqslant 2,$$

which cannot happen, since $l \geqslant 5$ and the points of $\sum_{i=1}^{7} Q_i$ are in a general position. Consequently, $\mathcal{L}$ is very ample with $h^1(C, \mathcal{L}) = 1$, whence $c := h^0(C, \phi^*\mathcal{N}) - h^0(C, \phi^*\mathcal{N}(-\sum_{i=1}^{7} P_i)) \geqslant 4$. According to Theorem 15, $\mathcal{L}$ fails to be normally generated, since the assumption $g > (n-1)(nd-6) + ng(C')$ implies $\deg \mathcal{L} = 2g - nd + 5 \geqslant \frac{3g}{2} - 1$ and so $h^1(C, \mathcal{L}^2(-\sum_{i=1}^{7} P_i)) = 0$.

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